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Note on Long-Step Predictor-Corrector Interior-Point Algorithm with Monteiro-Zhang Unified Search Directions

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Abstract

We present a long-step predictor-corrector interior-point algorithm for the monotone semidefinite linear complementarity problems using the Monteiro-Zhang unified search directions. Our algorithm is based on the long-step predictor-corrector interior-point algorithm proposed by Kojima, Shida and Shindoh using the Alizadeh-Haeberly-Overton search direction, although the AHO search direction does not belong to the MZ unified search directions in general.

1 Introduction

Recently, many authors have discussed generalization of interior-point algorithms for linear programming (LP) and monotone linear complementarity problems (LCPs) to the context of semidefinite programming (SDP) and monotone semidefinite linear complementarity problems (SDLCPs), see the list of references.

Let $\mathcal{M}$, $\mathcal{S}$, $\mathcal{S}_+$ and $\mathcal{S}_{++}$ be the class of $n \times n$ matrices, $n \times n$-symmetric matrices, positive semidefinite matrices in $\mathcal{S}$ and positive definite matrices in $\mathcal{S}$. For any two $p \times q$-matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, we denote $\mathbf{Tr}\ \boldsymbol{A}_1^\top \boldsymbol{A}_2$ by $\boldsymbol{A}_1 \bullet \boldsymbol{A}_2$ as an inner product. Let $\mathcal{F}$ be an $n(n+1)/2$-dimensional affine subspace of $\mathcal{S} \times \mathcal{S}$, and

$$\mathcal{F}_+ = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} : \boldsymbol{X} \succeq \boldsymbol{O}, \boldsymbol{Y} \succeq \boldsymbol{O}\}.$$

We are concerned with the Semidefinite Linear Complementarity Problem (SDLCP):

$$\text{Find an } (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+ \text{ such that } \boldsymbol{X} \bullet \boldsymbol{Y} = 0. \tag{1}$$

We call an $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}_+$ a feasible solution of the SDLCP (1). Throughout the paper, we assume the monotonicity of the affine subspace $\mathcal{F}$, *i.e.*,

$$(\boldsymbol{U}' - \boldsymbol{U}) \bullet (\boldsymbol{V}' - \boldsymbol{V}) \geq 0 \text{ for every } (\boldsymbol{U}, \boldsymbol{V}), (\boldsymbol{U}', \boldsymbol{V}') \in \mathcal{F}.$$

The monotone SDLCP was introduced in the paper [6] by Kojima, Shindoh and Hara as an extension of the monotone LCP, and discussed in [3, 5, 6, 15]. For positive semidefinite matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the complementarity condition $\boldsymbol{X} \bullet \boldsymbol{Y} = 0$ is equivalent to the condition $\boldsymbol{XY} = \boldsymbol{O}$. Therefore, to solve the monotone SDLCP (1), we numerically trace the perturbed system "central trajectory";

$$\mathcal{C} = \{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \text{ and } \boldsymbol{XY} = \mu \boldsymbol{I}\ (\mu > 0)\}$$

as $\mu \to 0$. Since our variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ are elements of the linear space of the symmetric matrices, we must choose a symmetric linearization for the asymmetric equation $\boldsymbol{XY} = \mu \boldsymbol{I}$.

To overcome the difficulty to choose a symmetric search direction, several ways are proposed by Alizadeh-Haeberly-Overton [1] and Kojima-Shindoh-Hara [6] (which include the search directions which are proposed by Helmberg-Rendl-Vanderbei-Wolkowicz [2] and Monteiro [7] and Nesterov-Todd [10]). As a generalization of Monteiro's approach, Zhang [16] introduced a general scheme, the so-called similar-symmetrization operator. Given a nonsingular $n \times n$-matrix $\boldsymbol{P}$, this operator is defined by

$$H_P(\boldsymbol{M}) = \frac{1}{2}[\boldsymbol{P}\boldsymbol{M}\boldsymbol{P}^{-1} + (\boldsymbol{P}\boldsymbol{M}\boldsymbol{P}^{-1})^{\mathsf{T}}] \quad \text{for } \forall \boldsymbol{M} \in \mathcal{M}. \tag{2}$$

The operator $H_P(\boldsymbol{M})$ is a projection from $\mathcal{M}$ to the subspace $\mathcal{S}$. Zhang [16] showed that $\boldsymbol{H}_P(\boldsymbol{M}) = \mu \boldsymbol{I} \Leftrightarrow \boldsymbol{M} = \mu \boldsymbol{I}$, for any nonsingular matrix $\boldsymbol{P}$, any matrix $\boldsymbol{M}$ with real spectrum, and any $\mu \in \mathcal{R}$. A perturbed Newton system using the operator leads to the following linear system;

$$\left.\begin{array}{l} (\boldsymbol{X} + d\boldsymbol{X}, \boldsymbol{Y} + d\boldsymbol{Y}) \in \mathcal{F}, \\ H_P(d\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{X}d\boldsymbol{Y}) = \beta\mu \boldsymbol{I} - H_P(\boldsymbol{X}, \boldsymbol{Y}), \end{array}\right\} \tag{3}$$

where $\beta \in [0, 1]$ is the centering parameter and $\mu = \mu(\boldsymbol{X}, \boldsymbol{Y}) \equiv (\boldsymbol{X} \bullet \boldsymbol{Y})/n$. The choices of $\boldsymbol{P} = \boldsymbol{X}^{-\frac{1}{2}}$ and $\boldsymbol{P} = \boldsymbol{Y}^{\frac{1}{2}}$ lead to the same formulas for two search directions proposed by Monteiro [7], which belong to the class of Kojima-Shindoh-Hara search directions proposed in a different formulation. The second search direction was also proposed in [2]. The choice of $\boldsymbol{P} = \boldsymbol{W}_{NT}^{\frac{1}{2}}$, where

$$\boldsymbol{W}_{NT} \equiv \boldsymbol{Y}^{\frac{1}{2}}(\boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{X}\boldsymbol{Y}^{\frac{1}{2}})^{-\frac{1}{2}}\boldsymbol{Y}^{\frac{1}{2}} = \boldsymbol{X}^{-\frac{1}{2}}(\boldsymbol{X}^{\frac{1}{2}}\boldsymbol{Y}\boldsymbol{X}^{\frac{1}{2}})^{\frac{1}{2}}\boldsymbol{X}^{-\frac{1}{2}},$$

leads to the Nesterov-Todd search direction [10], see also Sturm-Zhang [14] (the search direction also belongs to the class of KSH search directions, see [4]).

Recently, Monteiro-Zhang [9] proposed the class of nonsingular matrices

$$\begin{array}{rcl} \mathcal{P}(\boldsymbol{X}, \boldsymbol{Y}) & = & \{\boldsymbol{P} : \boldsymbol{P}^{\mathsf{T}}\boldsymbol{P} = \boldsymbol{W} \in \mathcal{S}_{++} \text{ such that } \boldsymbol{W}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}\boldsymbol{W}\} \\ & = & \{\boldsymbol{P} : \boldsymbol{P} \text{ is nonsingular and } \boldsymbol{P}\boldsymbol{X}\boldsymbol{Y}\boldsymbol{P}^{-1} \in \mathcal{S}\} \end{array} \tag{4}$$

and established the long-step interior-point algorithm using the search direction (3) corresponding to the class $\mathcal{P}(\boldsymbol{X}, \boldsymbol{Y})$. We note that their search directions only depends on $\boldsymbol{W}$ not on $\boldsymbol{P}$, *i.e.*, if $\boldsymbol{W} = \boldsymbol{P}_1^{\mathsf{T}}\boldsymbol{P}_1 = \boldsymbol{P}_2^{\mathsf{T}}\boldsymbol{P}_2$ then both corresponding systems have the same solution (Monteiro and Zhang [9] restrict $\boldsymbol{P}$ to a symmetric root of the matrix $\boldsymbol{W}$, for the simplicity of the argument). For $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$, $\boldsymbol{X}^{-\frac{1}{2}}$, $\boldsymbol{Y}^{\frac{1}{2}}$ and $\boldsymbol{W}_{NT}^{\frac{1}{2}}$ are in $\mathcal{P}(\boldsymbol{X}, \boldsymbol{Y})$. Alizadeh-Haeberly-Overton direction [1] can be described by $\boldsymbol{P} = \boldsymbol{I}$ in (3), but in general, $\boldsymbol{I}$ does not belong to $\mathcal{P}(\boldsymbol{X}, \boldsymbol{Y})$. For more details of the set $\mathcal{P}(\boldsymbol{X}, \boldsymbol{Y})$, see [9].

In this paper, we present a long-step predictor-corrector interior-point algorithm for the monotone Semidefinite Linear Complementarity Problems (SDLCPs) using the Monteiro-Zhang unified search directions. Nevertheless the Alizadeh-Haeberly-Overton search direction does not belong to the Monteiro-Zhang unified search directions in general, our algorithm is based on the paper [5] in which they use the AHO search direction.

In section 2, we present a long-step polynomial-time convergent predictor-corrector interior-point-algorithm for the monotone SDLCP. Local convergence of our algorithm is discussed in Section 3.

For the simplicity, we use the following notations;

$$\begin{aligned} \hat{X} = PXP^{\top}, \quad \hat{Y} = P^{-\top}YP^{-1}, \\ \widehat{dX} = PdXP^{\top}, \quad \widehat{dY} = P^{-\top}dYP^{-1}, \\ \hat{E} = \tfrac{1}{2}(\hat{Y} \otimes I + I \otimes \hat{Y}), \quad \hat{F} = \tfrac{1}{2}(\hat{X} \otimes I + I \otimes \hat{X}). \end{aligned}$$

2 Predictor-Corrector Interior-Point Algorithm

In this section, we present a long-step predictor-corrector interior-point algorithm for the monotone SDLCP (1) using the Monteiro-Zhang unified search directions.

It is easy to see that the linear system (3) gives well-defined search directions if we choose $P \in \mathcal{P}(X,Y)$;

Lemma 2.1. *For any $(X,Y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ and $P \in \mathcal{P}(X,Y)$, the system (3) has a unique solution.*

Proof: Since the first (feasibility) equation in (3) defines a maximal monotone affine subspace, we have only to show the strictly and maximal antitonicity of the second linear equation, which can be rewritten as follows;

$$\mathrm{vec}[dX]+(P\otimes P)^{-1}\hat{E}^{-1}\hat{F}(P^{-\top}\otimes P^{-\top})\mathrm{vec}[dY] = (P\otimes P)^{-1}\hat{E}^{-1}\mathrm{vec}[\beta\mu I - PXYP^{-1}].$$

By Proposition 3.2 of [9], we have that $\hat{E}\hat{F}$ is a symmetric positive definite matrix, thus so is $\hat{E}^{-1}\hat{F}$. Since P is nonsingular, we have that $(P \otimes P)^{-1}\hat{E}^{-1}\hat{F}(P^{-\top} \otimes P^{-\top}) = (P \otimes P)^{-1}\hat{E}^{-1}\hat{F}(P \otimes P)^{-\top}$ is positive definite. Therefore we conclude that the second equation defines a strictly and maximal antitone affine subspace. ∎

Throughout the paper, we use the following notation:

$$\begin{aligned} \rho \quad &: \quad \text{constant not less than } 1/n, \\ \mathcal{F}_0 \quad &= \quad \{(U',V')-(U,V) : (U,V),(U',V') \in \mathcal{F}\}, \text{(linearity space of } \mathcal{F}), \\ \mathcal{N}_W(\gamma,\tau) \quad &= \quad \left\{(X,Y) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : \begin{array}{l} \lambda_{min}(XY) \geq (1-\gamma)\tau, \\ X \bullet Y/n \leq (1+\rho\gamma)\tau \end{array}\right\} \\ & \quad\quad \text{for each } \gamma \in [0,1] \text{ and each } \tau \geq 0. \end{aligned}$$

Note that, for every $P \in \mathcal{P}(X,Y)$,

$$\lambda_{min}(XY) \;=\; \lambda_{min}(PXYP^{-1}) \;=\; \lambda_{min}(H_P(XY)).$$

By the definition, we see that

$$\begin{array}{ll} (1-\gamma)\tau \leq X \bullet Y/n & \text{if } (X,Y) \in \mathcal{N}_W(\gamma,\tau),\ \gamma \in [0,1] \text{ and } \tau \geq 0, \\ \mathcal{N}_W(0,\tau) \subset \mathcal{N}_W(\gamma,\tau) \subset \mathcal{N}_W(\gamma',\tau) & \text{if } 0 < \gamma < \gamma' \leq 1 \text{ and } \tau > 0. \end{array}$$

Note that $\mathcal{N}_W(0,\tau) = \{(X,Y) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : XY = \tau I\}$. Let $0 < \gamma < 1$. Then the set $\{(X,Y) \in \mathcal{N}_W(\gamma,\tau) : \tau > 0\}$ forms a (wide) neighborhood of the central surface $\{(X,Y) \in$

$\mathcal{S}_{++} \times \mathcal{S}_{++} : \boldsymbol{XY} = \tau \boldsymbol{I}$ for some $\tau > 0\}$. These sets serve as the admissible region in which we confine iterates $(\boldsymbol{X}^k, \boldsymbol{Y}^k)(k = 0, 1, 2, \cdots)$ of Algorithm 2.3 described below. More precisely, starting from a feasible point $(\boldsymbol{X}^0, \boldsymbol{Y}^0, \theta^0, \gamma^0) \in [\mathcal{F}(1) \cap (\mathcal{S}_{++} \times \mathcal{S}_{++})] \times \{1\} \times [0, 1)$, Algorithm 2.3 generates a sequence such that for every $k = 1, 2, \cdots$,

$$1 \geq \theta^k \geq 0, \gamma > \gamma^k \geq 0, \tag{5}$$
$$1 = \theta^0 > \theta^k > \theta^{k+1}, \tag{6}$$
$$(\boldsymbol{X}^k, \boldsymbol{Y}^k) \in \mathcal{N}_W(\gamma^k, \theta^k \mu^0) \cap \left[\mathcal{F} + \theta^k \left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\right)\right], \tag{7}$$
$$(\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c) \in \mathcal{N}_W(\gamma, \theta^{k+1} \mu^0) \cap \left[\mathcal{F} + \theta^{k+1} \left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\right)\right]. \tag{8}$$

Here $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ denotes an arbitrary pair of matrices in $\mathcal{F}$; in particular, we can take any feasible point of the SDLCP (1) for $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ when the SDLCP (1) has a feasible point. Note that

$$\begin{aligned} &\mathcal{F} + \theta \left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\bar{\boldsymbol{X}}', \bar{\boldsymbol{Y}}')\right) \\ &\quad = \mathcal{F} + \theta \left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\right) \\ &\qquad \text{for any } (\bar{\boldsymbol{X}}', \bar{\boldsymbol{Y}}'), (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}) \in \mathcal{F} \text{ and } \theta \in [0, 1]. \end{aligned}$$

Among the iterates $(\boldsymbol{X}^k, \boldsymbol{Y}^k, \boldsymbol{X}^k_c, \boldsymbol{Y}^k_c, \theta^k, \gamma^k)$, the triplet $(\boldsymbol{X}^k, \boldsymbol{Y}^k, \theta^k)$ is updated to $(\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c, \theta^{k+1})$ by the Predictor Step (Step 2), while the triplet $(\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c, \gamma^k)$ to $(\boldsymbol{X}^{k+1}, \boldsymbol{Y}^{k+1}, \gamma^{k+1})$ by the Corrector Step (Step 4). θ^{k+1} serves as a measure of both feasibility and optimality. Given an $\epsilon \geq 0$, the algorithm stops (at Step 3), when θ^{k+1} gets equal to or smaller than ϵ. In this case, we have an approximate solution $(\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c)$ of the SDLCP (1) such that

$$\left.\begin{array}{l} \epsilon \geq \theta^{k+1} \geq 0, \\ \boldsymbol{X}^k_c \succeq \boldsymbol{O}, \boldsymbol{Y}^k_c \succeq \boldsymbol{O}, \boldsymbol{X}^k_c \bullet \boldsymbol{Y}^k_c / n \leq (1 + \rho\gamma)\theta^{k+1}\mu^0, \\ (\boldsymbol{X}^k_c, \boldsymbol{Y}^k_c) \in \mathcal{F} + \theta^{k+1}\left((\boldsymbol{X}^0, \boldsymbol{Y}^0) - (\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})\right). \end{array}\right\} \tag{9}$$

We call ϵ an accuracy parameter.

Before we run Algorithm 2.3, we build up the hypothesis below. When the algorithm detects (at Step 1 or Step 3) that the hypothesis is false, it stops.

Hypothesis 2.2. *Let $\omega^* \geq 1$. There exists a solution $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ of the SDLCP (1) such that*

$$\omega^* \boldsymbol{X}^0 \succeq \boldsymbol{X}^* \textit{ and } \omega^* \boldsymbol{Y}^0 \succeq \boldsymbol{Y}^*. \tag{10}$$

Algorithm 2.3. [Long-Step Predictor-Corrector Interior-Point Algorithm]

Step 0: Choose an accuracy parameter $\epsilon \geq 0$, neighborhood parameter $\gamma^0 \in [0, 1)$ and an initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{N}_W(\gamma^0, \mu^0) \cap \mathcal{F}$, (we may choose any point $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ as an initial point, and let $\mu^0 = \boldsymbol{X}^0 \bullet \boldsymbol{Y}^0 / n$ and choose γ^0 so that $(1 - \gamma^0)\mu^0 < \lambda_{min}[\boldsymbol{X}^0 \boldsymbol{Y}^0]$).
(If $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{F}$, the SDLCP has a solution. Hence we may skip checking (12) in Step 1 and (14) in Step 3, since the SDLCP has a solution). Choose a neighborhood parameter $\gamma \in [\gamma^0, 1)$. Let $\theta^0 = 1$, $\sigma = 2\omega^*/(1 - \gamma) + 1$ and $k = 0$.

Step 1: If the inequality

$$\theta^k(\boldsymbol{X}^0 \bullet \boldsymbol{Y}^k + \boldsymbol{X}^k \bullet \boldsymbol{Y}^0) \leq \sigma \boldsymbol{X}^k \bullet \boldsymbol{Y}^k \tag{11}$$

does not hold then stop.

Step 2: (Predictor Step) Choose a matrix $\boldsymbol{P}^k \in \mathcal{P}(\boldsymbol{X}^k, \boldsymbol{Y}^k)$, and compute a solution $(d\boldsymbol{X}_p^k, d\boldsymbol{Y}_p^k)$ of the following equations;

$$\left.\begin{array}{l} (\boldsymbol{X}^k + d\boldsymbol{X}_p^k, \boldsymbol{Y}^k + d\boldsymbol{Y}_p^k) \in \mathcal{F}, \\ H_{P^k}(\boldsymbol{X}^k d\boldsymbol{Y}_p^k + d\boldsymbol{X}_p^k \boldsymbol{Y}^k) = -H_{P^k}(\boldsymbol{X}^k \boldsymbol{Y}^k) \end{array}\right\} \tag{12}$$

Let

$$\left.\begin{array}{rcl} \delta_p^k & = & \dfrac{\|\widehat{d\boldsymbol{X}}_p^k\|_F \|\widehat{d\boldsymbol{Y}}_p^k\|_F}{\theta^k \mu^0}, \\ \hat{\alpha}_p^k & = & \dfrac{2}{\sqrt{1 + 4\delta_p^k/(\gamma - \gamma^k)} + 1}, \\ \check{\alpha}_p^k & = & \max\left\{\alpha' \in [0,1] : \begin{array}{l} (\boldsymbol{X}^k + \alpha d\boldsymbol{X}_p^k, \boldsymbol{Y}^k + \alpha d\boldsymbol{Y}_p^k) \in \mathcal{N}_W(\gamma, (1-\alpha)\theta^k \mu^0) \\ \text{for every } \alpha \in [0, \alpha'] \end{array}\right\}. \end{array}\right\} \tag{13}$$

Choose a step length $\alpha_p^k \in [\hat{\alpha}_p^k, \check{\alpha}_p^k]$ (in Lemma 2.4, we will show $\hat{\alpha}_p^k \leq \check{\alpha}_p^k$).
Let $(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) = (\boldsymbol{X}^k, \boldsymbol{Y}^k) + \alpha_p^k(d\boldsymbol{X}_p^k, d\boldsymbol{Y}_p^k)$ and $\theta^{k+1} = (1 - \alpha_p^k)\theta^k$.

Step 3: If $\theta^{k+1} \leq \epsilon$ then stop. If the inequality

$$\theta^{k+1}(\boldsymbol{X}^0 \bullet \boldsymbol{Y}_c^k + \boldsymbol{X}_c^k \bullet \boldsymbol{Y}^0) \leq \sigma \boldsymbol{X}_c^k \bullet \boldsymbol{Y}_c^k \tag{14}$$

does not hold then stop.

Step 4: (Corrector Step) Choose a matrix $\boldsymbol{P}_c^k \in \mathcal{P}(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k)$, and compute a solution $(d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k)$ of the solution of equations;

$$\left.\begin{array}{l} (d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k) \in \mathcal{F}_0, \\ H_{P_c^k}(\boldsymbol{X}_c^k d\boldsymbol{Y}_c^k + d\boldsymbol{X}_c^k \boldsymbol{Y}_c^k) = \theta^{k+1}\mu^0 \boldsymbol{I} - H_{P_c^k}(\boldsymbol{X}_c^k \boldsymbol{Y}_c^k) \end{array}\right\} \tag{15}$$

Let

$$\left.\begin{array}{rcl} \delta_c^k & = & \dfrac{\|\widehat{d\boldsymbol{X}}_c^k\|_F \|\widehat{d\boldsymbol{Y}}_c^k\|_F}{\theta^{k+1}\mu^0}, \\ \hat{\alpha}_c^k & = & \begin{cases} \gamma/(2\delta_c^k) & \text{if } \gamma \leq 2\delta_c^k, \\ 1 & \text{if } \gamma > 2\delta_c^k, \end{cases} \\ \gamma^{k+1} & = & \begin{cases} \gamma(1 - \gamma/(4\delta_c^k)) & \text{if } \gamma \leq 2\delta_c^k, \\ \delta_c^k & \text{if } \gamma > 2\delta_c^k, \end{cases} \\ \hat{\gamma}^{k+1} & = & \min\left\{\gamma' \in [0,1] : \begin{array}{l} (\boldsymbol{X}_c^k + \alpha d\boldsymbol{X}_c^p, \boldsymbol{Y}^k + \alpha d\boldsymbol{Y}_c^k) \in \mathcal{N}_W(\gamma', \theta^{k+1}\mu^0) \\ \alpha \in [0,1] \end{array}\right\}. \end{array}\right\} \tag{16}$$

Choose a step length $\alpha_c^k \in [0,1]$ and γ^{k+1} such that

$$\left.\begin{array}{l} \hat{\hat{\gamma}}^{k+1} \leq \gamma^{k+1} \leq \check{\gamma}^{k+1}, \\ (\boldsymbol{X}_c^k + \alpha_c^k d\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k + \alpha_c^p d\boldsymbol{Y}_c^k) \in \mathcal{N}_W(\gamma^{k+1}, \theta^{k+1}\mu^0), \end{array}\right\} \tag{17}$$

(it will be shown in Lemma 2.4 that the pair of $\alpha_c^k = \hat{\alpha}_c^k$ and $\gamma^{k+1} = \check{\gamma}^{k+1}$ satisfies the relations above). Let $(\boldsymbol{X}^{k+1}, \boldsymbol{Y}^{k+1}) = (\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k) + \alpha_c^k(d\boldsymbol{X}_c^k, d\boldsymbol{Y}_c^k)$.

Step 5: Replace k by $k+1$. Go to Step 1.

Although the Alizadeh-Haeberly-Overton search direction does not belong to the Monteiro-Zhang unified search directions, Algorithm 2.3 is based on the paper [5], in which they use the AHO direction, with the different neighborhood $\widetilde{\mathcal{N}}_W(\gamma,\tau)$, where

$$\widetilde{\mathcal{N}}_W = \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : (\boldsymbol{X}\boldsymbol{Y} + \boldsymbol{Y}\boldsymbol{X})/2 \succ (1-\gamma)\tau \boldsymbol{I}, \boldsymbol{X} \bullet \boldsymbol{Y}/n \leq (1+\rho\gamma)\tau\}.$$

Let

$$\begin{array}{rcl} \breve{\alpha}_p^k & \equiv & \max\left\{\alpha' \in [0,1] : \begin{array}{l} (\hat{\boldsymbol{X}}^k + \alpha \widehat{d\boldsymbol{X}}_p^k, \hat{\boldsymbol{Y}}^k + \alpha \widehat{d\boldsymbol{Y}}_p^k) \in \widetilde{\mathcal{N}}_N(\gamma, (1-\alpha)\theta^k\mu^0) \\ \text{for every } \alpha \in [0,\alpha'] \end{array}\right\}, \\ \hat{\gamma}^{k+1} & \equiv & \min\left\{\gamma' \in [0,1] : \begin{array}{l} (\hat{\boldsymbol{X}}_c^k + \alpha \widehat{d\boldsymbol{X}}_c^p, \hat{\boldsymbol{Y}}^k + \alpha \widehat{d\boldsymbol{Y}}_c^k) \in \widetilde{\mathcal{N}}_N(\gamma', \theta^{k+1}\mu^0) \\ \alpha \in [0,1] \end{array}\right\} \end{array}$$

as in [5]. The following lemma gives a validity of Algorithm 2.3.

Lemma 2.4. *We have*

$$\hat{\alpha}_p^k \leq \breve{\alpha}_p^k \leq \check{\alpha}_p^k$$

and

$$\hat{\hat{\gamma}} \leq \hat{\gamma}^{k+1} \leq \check{\gamma}^{k+1}.$$

Proof: By Lemma 3.3 of [16], for any $(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$, any nonsingular matrix $\boldsymbol{P} \in \mathcal{P}(\boldsymbol{X},\boldsymbol{Y})$ and any nonsingular matrix $\boldsymbol{Q} \in \mathcal{M}$, we have

$$\lambda_{min}[H_Q(\boldsymbol{X}\boldsymbol{Y})] \leq \lambda_{min}[\boldsymbol{X}\boldsymbol{Y}] = \lambda_{min}[H_P(\boldsymbol{X}\boldsymbol{Y})].$$

Therefore, we conclude

$$(\hat{\boldsymbol{X}} + \alpha\widehat{d\boldsymbol{X}}, \hat{\boldsymbol{Y}} + \alpha\widehat{d\boldsymbol{Y}}) \in \widetilde{\mathcal{N}}_W(\gamma,\tau) \Rightarrow (\boldsymbol{X} + \alpha d\boldsymbol{X}, \boldsymbol{Y} + \alpha d\boldsymbol{Y}) \in \mathcal{N}_W(\gamma,\tau),$$

this implies that $\breve{\alpha}_p^k \leq \check{\alpha}_p^k$ and $\hat{\hat{\gamma}} \leq \hat{\gamma}^{k+1}$. By the analysis in the paper [5], we have $\hat{\alpha}_p^k \leq \breve{\alpha}_p^k$ and $\hat{\gamma}^{k+1} \leq \check{\gamma}^{k+1}$. ∎

The Monteiro-Zhang unified search direction can be rewritten by

$$\begin{array}{l} (\hat{\boldsymbol{X}} + \widehat{d\boldsymbol{X}}, \hat{\boldsymbol{Y}} + \widehat{d\boldsymbol{Y}}) \in \hat{\mathcal{F}}, \\ \hat{\boldsymbol{X}}\widehat{d\boldsymbol{Y}} + \widehat{d\boldsymbol{X}}\hat{\boldsymbol{Y}} + \hat{\boldsymbol{Y}}\widehat{d\boldsymbol{X}} + \widehat{d\boldsymbol{Y}}\hat{\boldsymbol{X}} = \beta\mu\boldsymbol{I} - (\hat{\boldsymbol{X}}\hat{\boldsymbol{Y}} + \hat{\boldsymbol{Y}}\hat{\boldsymbol{X}}), \end{array}$$

where $\hat{\mathcal{F}} = \{(\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) : (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F}\}$. Hence, we may interpret the Monteiro-Zhang search directions as "the symmetric linear transformation" + "the AHO search direction", see also [4].

$$\begin{array}{ccccc} \multicolumn{3}{c}{\textbf{Algorithm using MZ}} & & \textbf{Algorithm 2.3} \ [5] \\ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{N}_W(\gamma, \tau) & \rightarrow & (\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}) \in \widetilde{\mathcal{N}}_W(\gamma, \tau) & & (\boldsymbol{X}, \boldsymbol{Y}) \in \widetilde{\mathcal{N}}_W(\gamma, \tau) \\ & & \downarrow & & \downarrow \\ (\boldsymbol{X}^+, \boldsymbol{Y}^+) \in \mathcal{N}_W(\gamma', \tau') & \leftarrow & (\hat{\boldsymbol{X}}^+, \hat{\boldsymbol{Y}}^+) \in \widetilde{\mathcal{N}}_W(\gamma', \tau') & & (\boldsymbol{X}^+, \boldsymbol{Y}^+) \in \widetilde{\mathcal{N}}_W(\gamma', \tau'). \end{array}$$

Therefore, by Sections 3 and 4 of [5], it is easy to see the global convergence of the Algorithm 2.3;

Theorem 2.5. (Global Convergence Theorem):

(i) *Algorithm 2.3 consistently generates a sequence* $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k, \boldsymbol{X}_c^k, \boldsymbol{Y}_c^k, \theta^k, \gamma^k)\}$ *satisfying the relations (5)-(8).*

(ii) *If Algorithm 2.3 stops at Step 1 violating the inequality (11), then there is no solution of the SDLCP (1) satisfying (9).*

(iii) *If Algorithm 2.3 stops at Step 3 with* $\theta^{k+1} \leq \epsilon$, *then* $(\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k)$ *gives an approximate solution of the SDLCP (1) satisfying (9).*

(iv) *In Algorithm 2.3 stops at Step 3 violating the inequality (14), then there is no solution of the SDLCP 1 satisfying (10).*

(v) *If* $\epsilon > 0$, *Algorithm 2.3 stops in a finite number of iterations at either Step 1 or Step 3.* ∎

Remark 2.6. For the short-step algorithm, we only replace the neighborhood $\mathcal{N}_W(\gamma, \tau)$ with

$$\mathcal{N}_N(\gamma, \tau) = \left\{ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S}_{++} \times \mathcal{S}_{++} : \begin{array}{l} \|\boldsymbol{X}^{\frac{1}{2}} \boldsymbol{Y} \boldsymbol{X}^{\frac{1}{2}} - \tau \boldsymbol{I}\|_F \leq \gamma\tau, \\ \boldsymbol{X} \bullet \boldsymbol{Y}/n \leq (1 + \rho\gamma)\tau \end{array} \right\},$$

and let

$$\begin{aligned} \delta_p^k &= \frac{\|\widehat{d\boldsymbol{X}}_p \widehat{d\boldsymbol{Y}}_p\|_F}{\theta^k \mu^0}, \qquad \delta_c^k = \frac{\|\widehat{d\boldsymbol{X}}_p^k \widehat{d\boldsymbol{Y}}_p^k\|_F}{\theta^{k+1}\mu^0}, \\ \check{\alpha}_p^k &= \max \left\{ \alpha' \in [0,1] : \begin{array}{l} (\boldsymbol{X}^k + \alpha d\boldsymbol{X}_p^k, \boldsymbol{Y}^k + \alpha d\boldsymbol{Y}_p^k) \in \mathcal{N}_N(\gamma, (1-\alpha)\theta^k \mu^0) \\ \text{for every } \alpha \in [0, \alpha'] \end{array} \right\}, \\ \hat{\check{\gamma}}^{k+1} &= \min \left\{ \gamma' \in [0,1] : (\boldsymbol{X}_c^k + \alpha d\boldsymbol{X}_c^k, \boldsymbol{Y}_c^k + \alpha d\boldsymbol{Y}_c^k) \in \mathcal{N}_N(\gamma', \theta^{k+1}\mu^0) \right\}, \end{aligned}$$

in Algorithm 2.3. The validity of the short-step algorithm is easily derived by the same argument. ∎

In the rest of the section, we assume that the initial point $(\boldsymbol{X}^0, \boldsymbol{Y}^0)$ is a strictly feasible point of the SDLCP (1), to show the polynomial complexity of Algorithm 2.3. We first show the boundedness of numbers δ_p^k and δ_c^k. We give a bound by using the notation of the spectral condition number $\kappa(\boldsymbol{G}) \equiv \lambda_{max}[\boldsymbol{G}]/\lambda_{min}[\boldsymbol{G}]$ of $\boldsymbol{G} = \hat{\boldsymbol{E}}^{-1}\hat{\boldsymbol{F}}$ as in [9]. Let

$$\kappa_\infty \equiv \sup_k \kappa(\boldsymbol{G}^k) = \sup_k \frac{\lambda_{max}[(\hat{\boldsymbol{E}}^k)^{-1}\hat{\boldsymbol{F}}^k]}{\lambda_{min}[(\hat{\boldsymbol{E}}^k)^{-1}\hat{\boldsymbol{F}}^k]}. \tag{18}$$

Note that

$$\kappa_\infty \begin{cases} \leq \ \dfrac{n}{\gamma} & \text{if } \boldsymbol{P}^k = (\boldsymbol{X}^k)^{-\frac{1}{2}} \text{ or } (\boldsymbol{Y}^k)^{\frac{1}{2}} \text{ for } \forall k \\ = 1 & \text{if } \boldsymbol{P}^k = \boldsymbol{W}_{NT}^{\frac{1}{2}} \text{ for } \forall k, \end{cases}$$

(Theorem 6.2 of [9]). Recently Sheng, Potra and Ji [13] proposed a polynomial-time short-step primal-dual predictor-corrector infeasible-interior-point algorithm for the SDP with the additional assumption.

Lemma 2.7. *We have*

$$\delta_p^k \leq \frac{\sqrt{\kappa_\infty}}{2}(1+\rho\gamma)n \quad \textit{and} \quad \delta_c^k \leq \frac{\sqrt{\kappa_\infty}}{2}\frac{\gamma}{1-\gamma}(1+\rho\gamma)n.$$

Proof: By Lemma 6.2 of [9], we have

$$\|\widehat{d\boldsymbol{X}}_p^k\|_F \|\widehat{d\boldsymbol{Y}}_p^k\|_F \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}\boldsymbol{X}^k \bullet \boldsymbol{Y}^k \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}(1+\rho\gamma)\theta^k\mu^0 n$$

and

$$\|\widehat{d\boldsymbol{X}}_c^k\|_F \|\widehat{d\boldsymbol{Y}}_c^k\|_F \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}\frac{\gamma^k}{1-\gamma^k}\boldsymbol{X}_c^k \bullet \boldsymbol{Y}_c^k \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}\frac{\gamma}{1-\gamma}(1+\rho\gamma)\theta^{k+1}\mu^0 n.$$

Therefore we have

$$\delta_p^k \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}(1+\rho\gamma)n \ \text{ and } \ \delta_c^k \leq \frac{\sqrt{\kappa(\boldsymbol{G})}}{2}\frac{\gamma}{1-\gamma}(1+\rho\gamma)n,$$

thus we have the assertion. ∎

Next show the lower-bound of $\gamma - \check{\gamma}^k$.

Lemma 2.8. *For every $k = 1, 2, \cdots$,*

$$\gamma - \check{\gamma}^k \geq \frac{1}{2\sqrt{\kappa_\infty}}\frac{\gamma(1-\gamma)}{(1+\rho\gamma)n}(>0).$$

Proof: By Lemma 2.7, we have

$$\gamma - \check{\gamma}^k \geq \gamma - \gamma(1 - \frac{\gamma}{4\delta_c^k}) = \gamma^2/4\delta_c^k \geq \frac{\gamma(1-\gamma)}{2\sqrt{\kappa_\infty}(1+\rho\gamma)n}.$$

∎

Show the lower bound of the $\hat{\alpha}_p^k$.

Lemma 2.9. $\hat{\alpha}_p^k = \dfrac{1}{O(\sqrt{\kappa_\infty} n)}$.

Proof: By Lemma 2.8,

$$\frac{4\delta_p^k}{\gamma - \gamma^k} \geq \frac{4(1+\rho\gamma)^2 \kappa_\infty n^2}{\gamma(1-\gamma)} = O(\kappa_\infty n^2).$$

Therefore, we conclude

$$\hat{\alpha}_p^k = \frac{2}{\sqrt{1 + \frac{4\delta_p^k}{\gamma - \gamma^k}} + 1} = \frac{1}{O(\sqrt{\kappa_\infty} n)}.$$

∎

Theorem 2.10. *If we start at strictly feasible point* $(\boldsymbol{X}^0, \boldsymbol{Y}^0) \in \mathcal{N}_W(\gamma^0, \mu^0) \cap \mathcal{F}$, *Algorithm 2.3 terminates in at most* $O(\sqrt{\kappa_\infty} n \log(1/\epsilon))$.

Proof: By Lemma 2.9, we have that $\hat{\alpha}_p^k = \dfrac{1}{O(\sqrt{\kappa_\infty} n)}$. Therefore we may assume that $\hat{\alpha}_p^k \geq \dfrac{c}{\sqrt{\kappa_\infty} n}$. Hence

$$\theta^k = \theta^0 \Pi_{j=1}^{k-1}(1 - \alpha_p^j) \leq \theta^0 \left(1 - \frac{c}{\sqrt{\kappa_\infty} n}\right)^{k-1} \quad \text{for every } k = 1, 2, \cdots. \tag{19}$$

Therefore, we can conclude the assertion from the standard argument. ∎

3 Local Convergence.

In this section, we briefly discuss the superlinear convergence of Algorithm 2.3. We assume that there exists a solution of the monotone SDLCP (1) such that Hypothesis 2.2 holds.

Theorem 3.1. *If* $\delta_p^k = o(\theta^k)$ *(or* $O((\theta^k)^\nu)$ *for some* $\nu > 1$*), then the complementarity gap converges to zero superlinearly (or Q-order at least* ν*).*

Proof: We have

$$\begin{aligned} \theta^{k+1} = (1 - \alpha_p^k) &\leq 1 - \frac{2}{\sqrt{1 + \frac{4\delta_p^k}{\gamma - \gamma^k}} + 1} \\ &\leq \frac{\delta_p^k}{4(\gamma - \gamma^k)} \\ &= o(\theta^k) \qquad (\text{or } O((\theta^k)^\nu)) \end{aligned}$$

and this implies the complementarity gap converges to zero superlinearly (or order ν). ∎

Suppose that $(\boldsymbol{X}^*, \boldsymbol{Y}^*)$ is a solution of the monotone SDLCP (1). Since $\boldsymbol{X}^*$ and $\boldsymbol{Y}^*$ commute, there exists an orthogonal matrix $\boldsymbol{Q}$ such that

$$\boldsymbol{Q}^\top \boldsymbol{X}^* \boldsymbol{Q} = \begin{pmatrix} \Lambda_B & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \boldsymbol{Q}^\top \boldsymbol{Y}^* \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \Lambda_N \end{pmatrix},$$

where Λ_B and Λ_N are diagonal matrices. For $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} \times \mathcal{S}$, define

$$\boldsymbol{Q}^\top \boldsymbol{X} \boldsymbol{Q} = \underline{\boldsymbol{X}} = \begin{pmatrix} \underline{\boldsymbol{X}}_B & \underline{\boldsymbol{X}}_J \\ \underline{\boldsymbol{X}}_J^\top & \underline{\boldsymbol{X}}_N \end{pmatrix}, \ \boldsymbol{Q}^\top \boldsymbol{Y} \boldsymbol{Q} = \underline{\boldsymbol{Y}} = \begin{pmatrix} \underline{\boldsymbol{Y}}_B & \underline{\boldsymbol{Y}}_J \\ \underline{\boldsymbol{Y}}_J^\top & \underline{\boldsymbol{Y}}_N \end{pmatrix} \tag{20}$$

Define an affine subspace which contains the solution set of the monotone SDLCP;

$$\underline{\mathcal{F}} \equiv \left\{ (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{F} \cap (\mathcal{S} \times \mathcal{S}) : \underline{\boldsymbol{X}} = \begin{pmatrix} *_B & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \underline{\boldsymbol{Y}} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & *_N \end{pmatrix} \right\}.$$

Let $(\check{\boldsymbol{X}}^k, \check{\boldsymbol{Y}}^k)$ be the solution of the following minimization problem;

$$\min\{ \|\boldsymbol{P}^k(\boldsymbol{X}^k - \boldsymbol{X}')(\boldsymbol{Y}^k - \boldsymbol{Y}')(\boldsymbol{P}^k)^{-1}\|_F : (\boldsymbol{X}', \boldsymbol{Y}') \in \underline{\mathcal{F}}, \omega \boldsymbol{X}^0 \succeq \boldsymbol{X}', \omega \boldsymbol{Y}^0 \succeq \boldsymbol{Y}' \}. \tag{21}$$

Every accumulation point of the sequence $\{(\boldsymbol{X}^k, \boldsymbol{Y}^k)\}$ belongs to the feasible set of the above minimization problem (21) and the feasible set of (21) is bounded. Therefore $(\check{\boldsymbol{X}}^k, \check{\boldsymbol{Y}}^k)$ exists for each k. Let

$$\begin{gathered} \pi_X^k = \|\boldsymbol{P}^k \boldsymbol{X}^k (\boldsymbol{P}^k)^\top\|_F, \quad \pi_Y^k = \|(\boldsymbol{P}^k)^{-\top} \boldsymbol{Y}^k (\boldsymbol{P}^k)^{-1}\|_F, \\ \zeta_X^k = \|\boldsymbol{P}^k(\boldsymbol{X}^k - \check{\boldsymbol{X}}^k)(\boldsymbol{P}^k)^\top\|_F, \quad \zeta_Y^k = \|(\boldsymbol{P}^k)^{-\top}(\boldsymbol{Y}^k - \check{\boldsymbol{Y}}^k)(\boldsymbol{P}^k)^{-1}\|_F, \\ \eta_k = \|\boldsymbol{P}^k(\boldsymbol{X}^k - \check{\boldsymbol{X}}^k)(\boldsymbol{Y}^k - \check{\boldsymbol{Y}}^k)(\boldsymbol{P}^k)^{-1}\|_F. \end{gathered} \tag{22}$$

Theorem 3.2. *Suppose that* $\pi_X^k \pi_Y^k = O(\theta^k)$, $\pi_X^k \zeta_Y^k = O(\theta^k)$, $\pi_Y^k \zeta_X^k = O(\theta^k)$ *and* $\zeta_X^k \zeta_Y^k = o(\theta^k)$ *(or* $\eta^k = o(\theta^k)$ *for the short-step algorithm) Then the complementarity gap* $\boldsymbol{X}^k \bullet \boldsymbol{Y}^k / n$ *of generating sequence by Algorithm 2.3 converges to zero superlinearly. Moreover, if there exists a* $\nu > 1$ *such that* $\zeta_X^k \zeta_Y^k = O((\theta^k)^\nu)$ *(or* $\eta^k = O((\theta^k)^\nu)$ *for the short-step algorithm), then the convergence has Q-order at least* ν *in the sense that* $\theta^{k+1} = O(\theta^\nu)$.

Proof: Let $\Delta \boldsymbol{X} = d\boldsymbol{X}_p + (\boldsymbol{X} - \check{\boldsymbol{X}}), \Delta \boldsymbol{Y} = d\boldsymbol{Y}_p + (\boldsymbol{Y} - \check{\boldsymbol{Y}})$. It is easy to see that $(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y})$ is the solution of

$$\begin{aligned} &(\Delta \boldsymbol{X}, \Delta \boldsymbol{Y}) \in \mathcal{F}_0, \\ &H_P(\Delta \boldsymbol{X} \boldsymbol{Y} + \boldsymbol{X} \Delta \boldsymbol{Y}) = H_P((\boldsymbol{X} - \check{\boldsymbol{X}})(\boldsymbol{Y} - \check{\boldsymbol{Y}})) \\ &\begin{pmatrix} \boldsymbol{P}(\Delta \boldsymbol{X} \boldsymbol{Y} + \boldsymbol{X} \Delta \boldsymbol{Y})\boldsymbol{P}^{-1} + \boldsymbol{P}^{-\top}(\Delta \boldsymbol{Y} \boldsymbol{X} + \boldsymbol{Y} \Delta \boldsymbol{X})\boldsymbol{P}^\top \\ \quad = \boldsymbol{P}(\boldsymbol{X} - \check{\boldsymbol{X}})(\boldsymbol{Y} - \check{\boldsymbol{Y}})\boldsymbol{P}^{-1} + \boldsymbol{P}^{-\top}(\boldsymbol{Y} - \check{\boldsymbol{Y}})(\boldsymbol{X} - \check{\boldsymbol{X}})\boldsymbol{P}^\top \end{pmatrix}. \end{aligned}$$

By Lemma 3.1 of [5], we have

$$\|\boldsymbol{P} \Delta \boldsymbol{X} \boldsymbol{P}^\top\|_F \leq \frac{\|\boldsymbol{P} \boldsymbol{X} \boldsymbol{P}^\top\|_F \|H_P((\boldsymbol{X} - \check{\boldsymbol{X}})(\boldsymbol{Y} - \check{\boldsymbol{Y}}))\|_F}{(1-\gamma)\theta}$$

and

$$\|P^{-\top}\Delta Y P^{-1}\|_F \leq \frac{\|P^{-\top} Y P^{-1}\|_F \|H_P((X-\check{X})(Y-\check{Y}))\|_F}{(1-\gamma)\theta}.$$

Since

$$\|H_P((X-\check{X})(Y-\check{Y}))\|_F \leq \|P(X-\check{X})(Y-\check{Y})P^{-1}\|_F = \eta,$$

we have

$$\|P\Delta X P^\top\|_F \leq \frac{\|PXP^\top\|_F \|P(X-\check{X})(Y-\check{Y})P^{-1}\|_F}{(1-\gamma)\theta} = O(\pi_X \eta/\theta) = O(\pi_X \zeta_X \zeta_Y/\theta)$$

and

$$\|P^{-\top}\Delta Y P^{-1}\|_F \leq \frac{\|P^{-\top} Y P^{-1}\|_F \|P(X-\check{X})(Y-\check{Y})P^{-1}\|_F}{(1-\gamma)\theta} = O(\pi_Y \eta/\theta) = O(\pi_Y \zeta_X \zeta_Y/\theta).$$

Therefore we have

$$\begin{aligned}
\|d\hat{X}\|_F \|d\hat{Y}\|_F &= \|PdXP^\top\|_F \|P^{-\top} dY P^{-1}\|_F \\
&= \|P(\Delta X - (X-\check{X}))P^\top\|_F \|P^{-\top}(\Delta Y - (Y-\check{Y}))P^{-1}\|_F \\
&= \left(\frac{\pi_X \zeta_Y}{\theta} + 1\right)\left(\frac{\pi_Y \zeta_X}{\theta} + 1\right)\zeta_X \zeta_Y \\
&= O(\zeta_X \zeta_Y) \\
&= o(\theta) \qquad (\text{or } O(\theta^\nu)),
\end{aligned}$$

(or for the short-step algorithm, we have

$$\begin{aligned}
\|d\hat{X} d\hat{Y}\|_F &= \|P(\Delta X - (X-\check{X}))(\Delta Y - (Y-\check{Y}))P^{-1}\|_F \\
&\leq \|P\Delta X \Delta Y P^{-1}\|_F + \|P\Delta X(Y-\check{Y})P^{-1}\|_F \\
&\quad + \|P(X-\check{X})\Delta Y P^{-1}\|_F + \|P(X-\check{X})(Y-\check{Y})P^{-1}\|_F \\
&\leq \pi_X \pi_Y \eta^2/\theta^2 + \pi_X \eta \zeta_Y/\theta + \zeta_X \pi_Y \eta/\theta + \eta \\
&= O(\eta) \\
&= o(\theta) \qquad (\text{or } O(\theta^\nu))\ ).
\end{aligned}$$

By Theorem 3.5, we conclude the assertion. ∎

Remark 3.3. Potra and Sheng [12] prove the superlinear convergence of the short-step predictor-corrector infeasible-interior-point algorithm proposed by [3, 11] using the search direction given by $P^k = (X^k)^{-\frac{1}{2}}$ for every k, under conditions;

(i) the SDP problem has a strictly complementary solution
(ii) the size of the central path neighborhood approaches zero.

In their analysis [11, (3.9),(4.11) and (4.12)], they showed with the choice $P^k \equiv (X^k)^{-\frac{1}{2}}$, that

$$\begin{aligned}
\pi_X = O(1), \quad \pi_Y = O(\theta), \\
\zeta_X = O(1), \quad \zeta_Y = O(\theta).
\end{aligned}$$

by the strict complementarity condition (i). $\eta = o(\theta)$ by condition (ii). By their argument in [12], we can conclude the superlinear convergence of the short-step version of our algorithm (with $P^k = (X^k)^{-\frac{1}{2}}$ under the same conditions), though our algorithm (using $P^k = (X^k)^{-\frac{1}{2}}$ for every k) is slightly different from theirs. ∎

We need some preliminaries, for further analysis of the superlinear convergence of Algorithm 2.3;

Lemma 3.4. (Proposition 3.4 of [9]): *For any $\boldsymbol{P} \in \mathcal{P}(\boldsymbol{X},\boldsymbol{Y})$, there exists an orthogonal matrix $\boldsymbol{Q}_P$ such that*

$$\hat{\boldsymbol{X}} = \boldsymbol{Q}_P[\Lambda(\hat{\boldsymbol{X}})]\boldsymbol{Q}_P^{\mathsf{T}},\ \ \hat{\boldsymbol{Y}} = \boldsymbol{Q}_P[\Lambda(\hat{\boldsymbol{Y}})]\boldsymbol{Q}_P^{\mathsf{T}},\ \ \hat{\boldsymbol{X}}\hat{\boldsymbol{Y}}(=\hat{\boldsymbol{Y}}\hat{\boldsymbol{X}}) = \boldsymbol{Q}_P[\Lambda(\hat{\boldsymbol{X}}\hat{\boldsymbol{Y}})]\boldsymbol{Q}_P^{\mathsf{T}},$$

where $\Lambda(\boldsymbol{X})$ denotes a diagonal matrix with the eigenvalues of $\boldsymbol{X}$ on their diagonal elements. ∎

Monteiro and Zhang characterize the class of permissible matrices.

Lemma 3.5. (Theorem 3.1 of [9]): *Let $(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{S}_{++}\times\mathcal{S}_{++}$ and a fixed $\bar{\boldsymbol{P}} \in \mathcal{P}(\boldsymbol{X},\boldsymbol{Y})$ be given. Then any matrix $\boldsymbol{W} \in \mathcal{S}_{++}$ satisfying the equation $\boldsymbol{W}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}\boldsymbol{W}$ has the following representation in terms of $\boldsymbol{P}$;*

$$\boldsymbol{W} = \boldsymbol{W}(\bar{\boldsymbol{P}},\boldsymbol{T}) \equiv \bar{\boldsymbol{P}}^{\mathsf{T}}\boldsymbol{Q}_{\bar{P}}\boldsymbol{T}\boldsymbol{Q}_{\bar{P}}^{\mathsf{T}}\bar{\boldsymbol{P}}$$

where $\boldsymbol{Q}_P$ is an orthogonal matrix given in Proposition 3.4,

$$\boldsymbol{T} \equiv \mathbf{diag}(\boldsymbol{T}^{(1)},\cdots,\boldsymbol{T}^{(p)}), \boldsymbol{T}^{(i)} \in \mathcal{S}_{++}^{n_i}, i = 1,\cdots,p.$$

Moreover, the set $\{\boldsymbol{W} \in \mathcal{S}_{++} : \boldsymbol{W}\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{X}\boldsymbol{W}\}$ is a convex cone. ∎

Here we assume that the condition number $\kappa(\boldsymbol{T}^k)$ of $\boldsymbol{T}^k$ is bounded; $\mathcal{K} \equiv \sup\kappa(\boldsymbol{T}^k) = \sup\dfrac{\lambda_{max}(\boldsymbol{T}^k)}{\lambda_{min}(\boldsymbol{T}^k)} < \infty$, where $\boldsymbol{T}^k$ is given in Theorem 3.5 with $\bar{\boldsymbol{P}} = (\boldsymbol{X}^k)^{-\frac{1}{2}}$. By Theorem 6.3 of [9], $\kappa_\infty \le \frac{n}{1-\gamma}\mathcal{K}_\infty^2$. Then if $\mathcal{K}_\infty < \infty$ then $\kappa_\infty < \infty$. We use the following notations;

$$\begin{gathered}\bar{\pi}_X = \|\bar{\boldsymbol{P}}\boldsymbol{X}\bar{\boldsymbol{P}}^{\mathsf{T}}\|_F,\ \ \bar{\pi}_Y = \|(\bar{\boldsymbol{P}})^{-\mathsf{T}}\boldsymbol{Y}(\bar{\boldsymbol{P}})^{-1}\|_F,\\ \bar{\zeta}_X = \|\bar{\boldsymbol{P}}(\boldsymbol{X}-\check{\boldsymbol{X}})\bar{\boldsymbol{P}}^{\mathsf{T}}\|_F,\ \ \bar{\zeta}_Y = \|(\bar{\boldsymbol{P}})^{-\mathsf{T}}(\boldsymbol{Y}-\check{\boldsymbol{Y}})(\bar{\boldsymbol{P}})^{-1}\|_F,\\ \bar{\eta} = \|\bar{\boldsymbol{P}}(\boldsymbol{X}-\check{\boldsymbol{X}})(\boldsymbol{Y}-\check{\boldsymbol{Y}})(\bar{\boldsymbol{P}})^{-1}\|_F.\end{gathered}$$

Corollary 3.6. *Suppose that there exists a strictly complementary solution $(\boldsymbol{X}^*,\boldsymbol{Y}^*)$ of the monotone SDLCP (1) and the condition number $\kappa(\boldsymbol{T}^k)$ of $\boldsymbol{T}^k$ is bounded; $\mathcal{K} \equiv \sup\kappa(\boldsymbol{T}^k) < \infty$, where $\boldsymbol{T}^k$ is given in Theorem 3.5 with $\bar{\boldsymbol{P}}^k = (\boldsymbol{X}^k)^{-\frac{1}{2}}$. Assume that $\bar{\zeta}_X\bar{\zeta}_Y = o(\theta)$ (or $\bar{\eta} = o(\theta)$ for the short-step algorithm). Then the complementarity gap $\boldsymbol{X}^k \bullet \boldsymbol{Y}^k/n$ converges to zero superlinearly.*

Remark 3.7. The condition $\bar{\eta} = o(\theta)$ for the short-step algorithm holds if the sequence $\{(\boldsymbol{X}^k,\boldsymbol{Y}^k)\}$ generated by Algorithm 2.3 is tangentially convergent to the central trajectory; $\|\boldsymbol{X}^{\frac{1}{2}}\boldsymbol{Y}\boldsymbol{X}^{\frac{1}{2}} - \tau\boldsymbol{I}\| \le o(\tau)$ [12]. Sheng, Potra and Ji [13] showed the superlinear convergence of their short-step algorithm with the narrow neighborhood under the conditions (i) and (ii) using the same idea as follows. ∎

Proof of Corollary 3.6: From the definition, we have

$$\begin{aligned} \bar{\pi}_X &\equiv \|\bar{P}X\bar{P}^{\mathsf{T}}\|_F = \|I\|_F = O(1), \\ \bar{\pi}_Y &\equiv \|\bar{P}^{-\mathsf{T}}Y\bar{P}^{-1}\|_F = \|X^{\frac{1}{2}}YX^{\frac{1}{2}}\|_F = O(\theta). \end{aligned}$$

By the same argument of [11, (3.9), (4.11) and (4.12)], the existence of strictly complementary solution implies

$$\begin{aligned} \bar{\zeta}_X &\equiv \|\bar{P}(X-\check{X})\bar{P}^{\mathsf{T}}\|_F = O(1), \\ \bar{\zeta}_Y &\equiv \|\bar{P}^{-\mathsf{T}}(Y-\check{Y})\bar{P}^{-1}\|_F = O(\theta), \end{aligned}$$

for every $(X,Y) \in \mathcal{N}_W(\gamma,\tau)$. (Note that these estimations are valid for the wide neighborhood $\mathcal{N}_W(\gamma,\tau)$.) Since P is described by $T^{\frac{1}{2}}Q_{\bar{P}}^{\mathsf{T}}\bar{P}$, we have

$$\begin{aligned} \pi_X &\equiv \|PXP^{\mathsf{T}}\|_F \le \|T^{\frac{1}{2}}\|\|Q_{\bar{P}}^{\mathsf{T}}\|\|\bar{P}X\bar{P}^{\mathsf{T}}\|_F\|Q_{\bar{P}}\|\|T^{\frac{1}{2}}\| \\ &= \|T^{\frac{1}{2}}\|^2\bar{\pi}_X \le \|T^{\frac{1}{2}}\|^2O(1), \\ \pi_Y &\equiv \|P^{-\mathsf{T}}YP^{-1}\| \le \|T^{-\frac{1}{2}}\|\|Q_{\bar{P}}^{\mathsf{T}}\|\|\bar{P}^{-1}Y\bar{P}^{-1}\|_F\|\|Q_{\bar{P}}\|\|T^{-\frac{1}{2}}\| \\ &= \|T^{-\frac{1}{2}}\|^2\bar{\pi}_Y \le \|T^{-\frac{1}{2}}\|^2O(\theta), \\ \zeta_X &\equiv \|P(X-\check{X})P^{\mathsf{T}}\|_F \le \|T^{\frac{1}{2}}\|Q_{\bar{P}}^{\mathsf{T}}\|\|\bar{P}(X-\check{X})\bar{P}^{\mathsf{T}}\|_F\|Q_{\bar{P}}\|\|T^{\frac{1}{2}}\| \\ &= \|T^{\frac{1}{2}}\|^2\bar{\zeta}_X \le \|T^{\frac{1}{2}}\|^2O(1), \\ \zeta_Y &\equiv \|P^{-\mathsf{T}}(Y-\check{Y})P^{-1}\|_F \le \|T^{-\frac{1}{2}}\|\|Q_{\bar{P}}^{\mathsf{T}}\|\|\bar{P}^{-\mathsf{T}}(Y-\check{Y})\bar{P}^{-1}\|_F\|Q_{\bar{P}}\|\|T^{-\frac{1}{2}}\| \\ &= \|T^{-\frac{1}{2}}\|^2\bar{\zeta}_Y \le \mathcal{K}_\infty O(\theta), \\ \zeta_X\zeta_X &\equiv \|P(X-\check{X})P^{\mathsf{T}}\|_F\|P^{-\mathsf{T}}(Y-\check{Y})P^{-1}\|_F \le \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2\bar{\zeta}_X\bar{\zeta}_Y \\ &= \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2\bar{\zeta}_X\bar{\zeta}_Y \le \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2o(\theta) \\ (\,\eta &\equiv \|P(X-\check{X})(Y-\check{Y})P^{-1}\|_F \le \|T^{\frac{1}{2}}\|\|Q_{\bar{P}}^{\mathsf{T}}\|\|\bar{P}(X-\check{X})(Y-\check{Y})\bar{P}^{-1}\|_F\|Q_{\bar{P}}\|\|T^{-\frac{1}{2}}\| \\ &= \|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|\bar{\eta} \le \|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|o(\theta),\,). \end{aligned}$$

Hence we have

$$\begin{aligned} \pi_X\pi_Y &\le \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2O(\theta) &&\le \mathcal{K}_\infty O(\theta), \\ \pi_X\zeta_Y &\le \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2O(\theta) &&\le \mathcal{K}_\infty O(\theta), \\ \pi_Y\zeta_X &\le \|T^{-\frac{1}{2}}\|^2\|T^{\frac{1}{2}}\|^2O(\theta) &&\le \mathcal{K}_\infty O(\theta), \\ \zeta_X\zeta_Y &= \|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2o(\theta) &&\le \mathcal{K}_\infty o(\theta), \\ (\,\eta &= \|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|o(\theta) &&\le \sqrt{\mathcal{K}_\infty}o(\theta).\,). \end{aligned}$$

Therefore, by Theorem 3.2, we conclude the assertion. Here we use $\|Q_{\bar{P}}\| = \|Q_{\bar{P}}^{\mathsf{T}}\| = 1$ and

$$\|T^{\frac{1}{2}}\|^2\|T^{-\frac{1}{2}}\|^2 = \|T\|\|T^{-1}\| = \frac{\lambda_{max}(T)}{\lambda_{min}(T)} \le \mathcal{K}_\infty.$$

∎

Remark 3.8. If one can show the superlinear convergence of long-step path-following algorithm using one specific choice of a sequence $\{\bar{P}^k \in \mathcal{P}(X^k,Y^k)\}$ of matrices by showing

$$\bar{\pi}_X\bar{\pi}_Y = O(\tau),\ \bar{\zeta}_X\bar{\pi}_Y = O(\tau),\ \bar{\pi}_Y\bar{\zeta}_X = O(\tau),\ \bar{\zeta}_X\bar{\zeta}_Y = o(\tau),$$

then, using the same idea as in Corollary 3.6, we can conclude the superlinear convergence of long-step path-following algorithm using more general sequence $\{P^k \in \mathcal{P}(X^k,Y^k)\}$ under same boundedness condition of T^k. ∎

4 Concluding Remarks

In this paper, we present a long-step predictor-corrector path-following interior-point algorithm for monotone semidefinite linear complementarity problems using the Monteiro-Zhang unified search direction.

If we choose the strictly feasible initial point, the complementarity gap polynomially converges to zero by Theorem 2.10 using the (wide) neighborhood. Conversely, Kojima,Shida and Shindoh [5] proposed the long-step predictor-corrector (infeasible)-interior-point algorithm using AHO direction, which generates the sequence such that the complementarity gap quadratically converges to zero under the strict complementarity condition. Here, we consider to apply our algorithm to the monotone diagonal SDLCP (which is equivalent to the monotone LCP). Since each Monteiro-Zhang unified search direction is equal to the AHO direction in this case, we can conclude that Algorithm 2.3 is the long-step globally polynomial-time, locally quadratically convergent predictor-corrector (but feasible) interior-point algorithm, without any additional estimation for the diagonal SDLCPs.

Recently Monteiro [8] present the polynomial time convergence (independent of the condition number $\kappa(\boldsymbol{G})$) of the short-step interior-point algorithm (for the SDP) using more general class of search directions (3), which includes the AHO search direction and the MZ unified search directions.

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